Field of Fractions

Q/ : How do we go from $\mathbb{Z}$ to $\mathbb{Q}$?

$\frac{a}{b} \in \mathbb{Q} \Rightarrow a, b \in \mathbb{Z}$, $b \neq 0$

$\frac{a}{b} = \frac{c}{d} \iff ad - bc = 0$

Hence $\mathbb{Q}$ = ordered pairs (a,b) where

1/ $a, b \in \mathbb{Z}$

2/ $b \neq 0$

3/ $(a,b) \sim (c,d) \iff ad - bc = 0$

give the same element of $\mathbb{Q}$

$ab = 0_R$
$\Rightarrow a = 0_R$ or $b = 0_R$

Definition Let R be an integral domain.

We define the relation $\sim$ on $R \times (R \setminus \{0_R\})$ as follows :

$$(a,b) \sim (c,d) \iff ad - bc = 0_R$$

must be integral domain

Proposition $\sim$ is an equivalence relation on $R \times (R \setminus \{0_R\})$.

Proof : Tedious exercise. Do it so you understand why R must be an integral domain.

Field of Fractions of R

Definition $\mathrm{Frac}(R) = R \times (R \setminus \{0_R\}) / \sim$

collection of equivalence classes under $\sim$

Notation : $[(a,b)] = \frac{a}{b}$ ← usual fraction notation

Examples $\text{Frac}(\mathbb{Z}) = \mathbb{Q}$

square bracket   curly bracket

$$\text{Frac}(R[x_1,\dots,x_n]) = R(x_1,x_2,\dots,x_n)$$

Fact : Frac(R) is a field under the following + and ×

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Familiar Properties :

$\frac{a}{b} = 0_{\text{Frac}(R)} \iff a = 0_R$

1/ $0_{\text{Frac}(R)} = \frac{0_R}{b}$ for any $b \neq 0_R$

2/ $1_{\text{Frac}(R)} = \frac{1_R}{1_R} = \frac{b}{b}$ for any $b \neq 0_R$

3/ $-\left(\frac{a}{b}\right) = \frac{-a}{b} = \frac{a}{-b}$, $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$ ← This is why it's a field

4/ $\frac{a}{1_R} = \frac{c}{1_R} \iff a = c \quad \forall a,c \in R$

$\Rightarrow$ $\phi : R \longrightarrow \text{Frac}(R)$ is an injective homomorphism
$\quad a \longrightarrow \frac{a}{1_R}$

$\Downarrow$

R isomorphic to a subring of Frac(R)

We generally write $R \subset \text{Frac}(R)$

5/ R a field $\Rightarrow$ ϕ an isomorphism so

$R \cong \text{Frac}(R)$ ← $\frac{a}{b} = \frac{\left(\frac{a}{b}\right)}{1_R}$